Analysis of the Instantaneous Estimate of Autocorrelation

Alec Rogers, *Member, IEEE*
alec@ece.pdx.edu

***Abstract*— This paper is an analysis of the Instantaneous Estimate of Autocorrelation (IEAC), which is defined as a signal formed by taking the product of the current value of a signal and the vector composed of its delayed values. The mean of this signal is the unbiased estimate of the autocorrelation of the original signal. Here, we explore several useful statistics associated with this signal, and suggest applications where they may be put to good use. Experiments are performed which demonstrate the beneficial properties of preserving more of this signal then is contained in a simple point estimate of its mean (i.e. the estimated autocorrelation).**

***Index Terms*— Autocorrelation, nonstationary, time-varying filters, PCA.**

I. Introduction

THIS paper is an analysis of a signal which we refer to as the Instantaneous Estimate of Autocorrelation (IEAC) signal. This is a vector valued, second order signal which is defined in terms of a scalar valued, first order signal (which we will consider to be a time series). It is a signal whose vector coefficients at time n correspond to a product of the signal with itself at lag ℓ, and whose mean over a length N represents an estimate of the autocorrelation of the original signal.

Approaches to characterizing data that make use of a single estimate of the autocorrelation of the signal are often (at least implicitly) assuming that the signal is stationary (at least in the wide sense); otherwise their use of this statistic would be meaningless. In many cases, this assumption does not hold exactly; in those cases, local stationarity (a much less strict condition) is often assumed in order to make use of second order statistics. It is then possible to model the signal as a series of time-limited signals, each of which is stationary over its duration. This approach is taken in many audio encoding schemes, where sound may be assumed to have constant spectral characteristics over a short interval.

We consider the reduction of the autocorrelation signal to a point estimate to be, in general, harmful. Even short-duration frames which assume stationarity within their restricted temporal interval are not, perhaps, motivated by the data itself, but rather by the benefits which are conferred by 'blocking' the signal. In this paper, we explore the IEAC signal in order to see how we might characterize this signal with several parameters.

Our work in this paper is intended, to some extent, to lay a foundation for time-varying filters which are based on the time-varying autocorrelation. By characterizing the changes of the IEAC signal over time, we can develop filters which are better suited to signals which also change over time. Here, we examine some of the problems associated with filters based on the stationary autocorrleation, and suggest ways to go beyond these types of filters.

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A. Rogers is with the Portland State University ECE department

II. The Instantaneous Estimate of Autocorrelation Defined

Given a signal $x(n)$ of length N, we define the IEAC ($\mathbf{x_c}(n)$) as follows:

$$\mathbf{x_c}(n) \triangleq [x(n)^2 \ x(n)x(n-1) \ ... \ x(n)x(n-M)] \quad (1)$$

Where M denotes the number of lagged values of the vector. We assume that $x(n) = 0$ for $n < 0$, thus we form a vector representing the signal as $\mathbf{x}_c$ which is of dimension $N \times (M+1)$.

A. The First Moment

The first moment of this series may be estimated as follows:

$$\mu_{\mathbf{x_c}} \triangleq \frac{1}{N}\sum_{i=0}^{N-1} \mathbf{x_c}(i) \quad (2)$$

The use of this estimator to determine the mean of the IEAC signal has a nice property: it is identical to the vector composed of the biased estimates of autocorrelation ($\hat{r}_x(\ell)$, [1]) of the original signal at lags $0 \leq \ell \leq M$ ($\hat{r}_x(\ell)$):

$$\hat{r}_x(\ell) \triangleq \frac{1}{N}\sum_{n=|\ell|}^{N-1} x(n)x(n-|\ell|) \quad (3)$$

$$\mu_{\mathbf{x_c}} = [\hat{r}_x(0) \ \hat{r}_x(1) \ \dots \ \hat{r}_x(M)] \quad (4)$$

Accordingly, we may form the estimate of normalized autocorrelation by normalizing $\mu_{\mathbf{x_c}}$ by its first element:

$$\hat{\rho}_x \triangleq \frac{1}{\mu_{x_c}[0]}\mu_{x_c} \quad (5)$$

The normalized mean of the IEAC gives us a commonly used estimate of the Autocorrelation Function (ACF), and thus may be viewed as a different method of computation which arrives at familiar results. What is significant in the change of notation is that the autocorrelation is now viewed as a vector-valued signal that is a function of time (which we may easily transform into a time-varying, all-pole filter). Thus, an advantage of $\mathbf{x}_c$ is that we may study it as a random vector itself. However, we *don't want* to reduce the signal to its mean;

this amounts to the assumption that the IEAC is stationary of the first order, which in turn implies that the original signal is stationary of the second order. We are interested in how we can characterize the signal before we (potentially) decide to make this assumption.

There is a problem, however, in using the instantaneous estimates of autocorrelation directly; they vary to a large degree due to the presence of noise in the signal. In order to eliminate some of this variance, we will window the elements of the IEAC at all lags with a (normalized) Hanning window. Thus, we are assuming that the autocorrelation of the signal will change slowly. It is important to note in this respect that windowing the IEAC is not equivalent to windowing the orignal signal and then computing the autocorrelation estimates. Since the only window used for the computation of the IEAC is the window of observation, we are computing our estimate of autocorrelation using a rectangular window and then averaging this signal across time at each lag (with the normalized Hanning window). Hence, our estimate will only be biased at the beginning of the IEAC (i.e. at times $n < M$, where M denotes the number of lags in the IEAC signal). This is where we are forced to use zeros for unknown values of x ($x(n)$ for $n < 0$). After these $M-1$ initial samples, the normalized windowing of the IEAC does not bias the signal.

B. The Second (Central) Moment

Variance, as the IEAC is not a scalar signal, can be estimated by projection onto an arbitrary vector of the subspace. In order to find the vector corresponding to the maximal variance, we use Principal Component Analysis (PCA). Note that this approach is different from the one suggested by formulating the ACF as a function of lag; in that case, we are more prone to look at the variance of each correlation coefficient independently.

By projecting the IEAC onto the first principal component of its variance, we obtain a scalar-valued function that indicates how the autocorrelation vector evolves over time. Inherent in the vector formulation of variance is the assumption that the autocorrelation values at different lags will not change independently of each other. This does not mean that the autocorrelations at different lags cannot change in different directions; we only require that they change at the same rate. If they did change at different rates, extraction of the principal component of their (joint) variance would probably not be a good characterization of their change with respect to time.

In order to do PCA, we formulate the autocorrelation matrix of $\mathbf{x_c}$ and decompose it using eigendecomposition:

$$\mathbf{C_{x_c}} \triangleq \frac{1}{N}\mathbf{x_c}^H\mathbf{x_c} \quad (6)$$

$$= \mathbf{Q\Lambda Q}^H = [\mathbf{q}_0\mathbf{q}_1\ldots\mathbf{q}_M]\mathbf{\Lambda}[\mathbf{q}_0\mathbf{q}_1\ldots\mathbf{q}_M]^H \quad (7)$$

The matrix $\mathbf{C_{x_c}}$ has entries which are estimates of the fourth order moments of the original signal, which may be expressed as:

$$E\{x(n)x(n+k)x(n+\ell)x(n+m)\} \quad (8)$$

In our case, $k=0$ and $\ell, m \sim [0, M]$.

C. Spectral Interpretations

If we have an ACF that changes with time, then we also have a Power Spectral Density (PSD) that changes with time, since the PSD of a signal is defined as the fourier transform of its autocorrelation. This analysis requires us to use the Short Time Fourier Transform to define the time-varying PSD.

If we assume that the model is stationary, then eigenvalue spread is indicative of the spectral dynamic range of the signal ([1]), according to:

$$\min_{\omega} R(e^{j\omega}) \leq \lambda_i \leq \max_{\omega} R(e^{j\omega}) \quad (9)$$

If, on the other hand, we do not make the assumption of stationarity, then the spectral dynamic range may in fact be attributable to the non-stationarity of the signal. For example, a temporal decomposition of the signal may yield sequences which taken separately do not have a wide spectral dynamic range. This would imply that each sub-sequence has less variation in its eigenvalues, which is beneficial for signal compression.

In order to reduce the variance of the PSD estimate, we window the IEAC across lags: this is equivalent to using the Blackman-Tukey method of variance reduction for spectral estimation. If we determine that the PSD is stationary, we can time average the PSDs generated by the IEAC (according to the Walsh-Bartlett method). Note that using both of these methods probably introduces too much bias into our estimation.

III. Methodology

For the first experiment, we analyze two signals which are generated from two sources, each of which is filtered pseudo-random noise. The sources (s_1, s_2) and signals (x_1, x_2) are defined as follows:

$$\begin{aligned} s_1(n) &\triangleq 1.0w(n) + 0.4w(n-1) - 0.4w(n-2) \\ s_2(n) &\triangleq 1.0w(n) + 0.4w(n-1) + 0.4w(n-2) \\ x_1(n) &\triangleq s_1(n) \\ x_2(n) &\triangleq \begin{cases} s_1(n) & 0 \leq n < \frac{N}{2} \\ s_2(n) & \frac{N}{2} \leq n < N \end{cases} \end{aligned}$$

where $w(n)$ denotes white Gaussian noise with zero mean and unit variance, and the length of the signal is $N = 4096$. From the analysis of the two signals x_1 and x_2, we see that they have the following ACF:

$$\begin{aligned} \rho_{x_1} &= [1\ 0.4\ -0.4] \\ \rho_{x_2} &= [1\ 0.4\ 0] \end{aligned}$$

We note that the second coefficient of the second signal is an average of $[0.4, -0.4]$, since we are taking the time average over all N.

Our analysis of these signals proceeds by first forming the IEAC signal at lags $[0, 32]$. Next, we form the autocorrelation matrix of the IEAC signal, from which we can determine the principal axes of signal variation. In order to plot the data (in two dimensions), we use the IEAC at lags 1 and 2 as our axes, and plot the eigenvectors corresponding to these two dimensions as scaled in magnitude by their respective eigenvalues. We then compare this to the residual of the signal after we remove the prediction based on our estimate of the (stationary) autocorrelation (we use an optimal all-pole predictor of order two). Finally, we show the residual signal as projected on the first principal component of the signal, which corresponds to the temporal evolution of the IEAC on the axis of its greatest variance.

IV. RESULTS

The sample data for $x_2(n)$ is shown in Figure 1. The first signal is not plotted, but obviously looks similar to the first half of the $x_2(n)$. The IEAC signal is plotted as a function of time for lags $[1, 32]$ in Figure 2 (we have omitted the case of lag 0 since it dominates the plot of signal autocorrelation). We note that the plot of the autocorrelation at lag 2 changes abruptly in the middle of the signal, corresponding to the change of autocorrelation present in the definition of the signal.

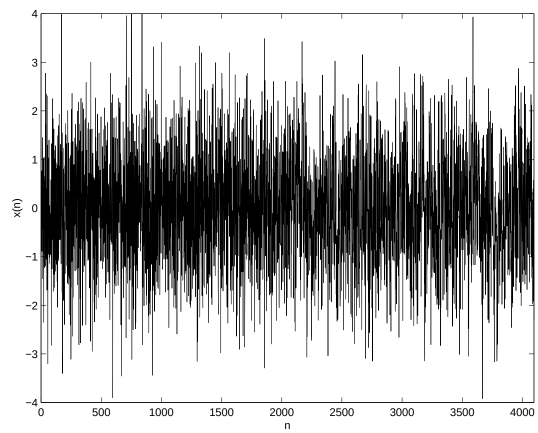

Fig. 1. $x_2(n)$, the original signal

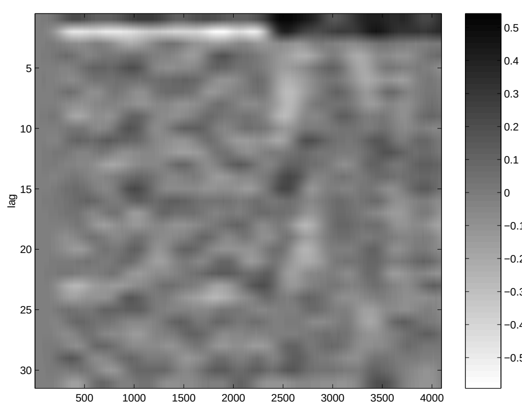

Fig. 2. $\mathbf{x_{2c}}$: Lags 1-32 of the IEAC signal.

Estimates of the (normalized) autocorrelation and the eigenmatrices of the two signals are shown in Table I. We observe that our estimates of autocorrelation are biased toward zero compared to the actual values. The eigenvalue spread is 0.1827 for the first signal, and 0.5638 for the second.

The plot of every eighth point of the IEAC of $x_2(n)$ is shown in Figure 3. Also shown on the plot are axes which correpond to our estimate of the autocorrelation of the signal (which is the mean of the points). The eigenvectors $(\mathbf{q_i})$ are depicted with heavy black lines, after having been multiplied by their respective eigenvalues (λ_i).

TABLE I

SIGNAL STATISTICS (ORIGINAL SIGNAL)

	$x_{c1}(n)$	$x_{c2}(n)$
$\hat{\mu}$	[1.0 0.1505 -0.3203]	[1.0 0.2605 -0.0184]
$\mathbf{Q}$	$\begin{bmatrix} 0.9741 & 0.2259 \\ -0.2259 & 0.9741 \end{bmatrix}$	$\begin{bmatrix} 0.7422 & 0.6702 \\ -0.6702 & 0.7422 \end{bmatrix}$
λ	$\begin{bmatrix} 0.9907 & 0 \\ 0 & 1.1734 \end{bmatrix}$	$\begin{bmatrix} 0.7589 & 0 \\ 0 & 1.2957 \end{bmatrix}$

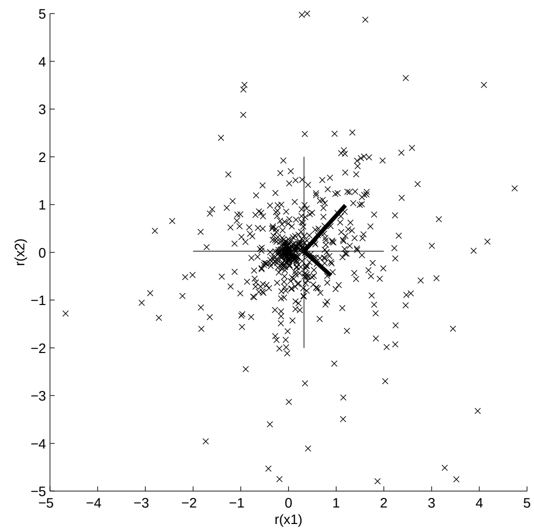

Fig. 3. $\mathbf{x_{2c}}$: points projected on the $\ell = [1, 2]$ subspace.

In table II, we see the autocorrelation and eigenmatrices corresponding to the residual signals, which are formed by removing the component that could be predicted by an all-pole predictor. As expected, the (stationary) autocorrelation is very close to zero. The eigenvalue spread, however, has been reduced to 0.0964 in the first case, and 0.1487 in the second. The direction of maximal variance for the second signal is in the direction of the autocorrelation at lag 2, which is what we would expect from the non-stationary definition of the signal. The plot of the IEAC points for the second sequence is shown in Figure 4.

TABLE II

SIGNAL STATISTICS (AFTER REMOVING PREDICTION)

	$x_{c1}(n)res$	$x_{c2}(n)$ res
$\hat{\mu}$	[1.0 0.0461 -0.0664]	[1.0 0.0099 -0.0049]
$\mathbf{Q}$	$\begin{bmatrix} -0.8725 & 0.4887 \\ -0.4887 & -0.8725 \end{bmatrix}$	$\begin{bmatrix} -0.9967 & -0.0811 \\ 0.0811 & 0.9967 \end{bmatrix}$
λ	$\begin{bmatrix} 1.0230 & 0 \\ 0 & 0.9266 \end{bmatrix}$	$\begin{bmatrix} 0.9990 & 0 \\ 0 & 1.1477 \end{bmatrix}$

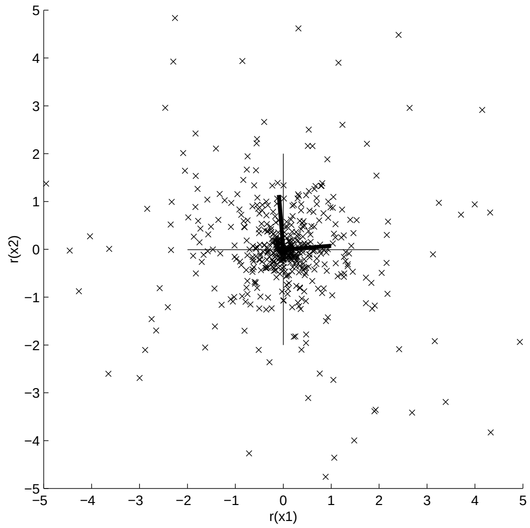

Fig. 4. $\mathbf{x}_{2c}$ residual: points projected on the $\ell = [1, 2]$ subspace after filtering with an all-pole filter

The projection of the residual of $\mathbf{x}_{\mathbf{c_2}}$ onto its first principal component is shown in Figure 5. Since this principal component points almost exactly in the direction of the autocorrelation at lag 2 (we are off by less than 5 degrees), we have also plotted the actual autocorrelation at lag 2.

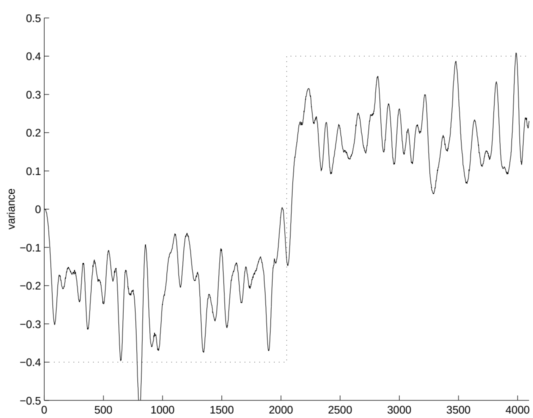

Fig. 5. $PC_{x_{2_c}}(1)$: Variation of the IEAC projected on the first Principal Component of the $(r_x(1), r_x(2))$ subspace. The dotted line shows the true autocorrelation at lag 2.

V. Discussion

The eigenvalue spread for both signals is reduced by all-pole filtering. For the first signal, we note that the eigenvalues become close to 1, which means that our residual consists primarily of white noise (as far as we can determine using second-order methods). This is not true to the same degree for the second signal. Further, we see by projecting the IEAC onto the first principal component (which is in the direction of $r_x(2)$) that the signal changes mean at half its length. This should not be surprising, since that is exactly what the data itself does. This suggests that if we are frame blocking the signal, we should make our frame boundary in the middle of the signal, which would significantly reduce the variation in the eigenvalues.

By plotting the IEAC on its principal axis subspace, we have seen that we may observe the variation of the signal's autocorrelation coefficients. For the contrived signal presented here, we see a simple level change, which occurs in the direction of only one of the autocorrelation lags (i.e. lag 2). In general, however, we may have a change which is more complicated than a level change (e.g. one which is best described by a polynomial). The change might also take place across several lags, instead of just one.

VI. Conclusion

If we assume that the signal does not come from a process which is WSS, then the IEAC signal does not have a stationary mean. Hence, we must find a way to characterize the behavior of IEAC signals which have time-varying properties (i.e. we should not use only the mean).

We suggest three ways in which a non-stationary signal may be represented in terms of changing filter parameters. The first method is to assume that the model is stationary over a small interval. This assumption amounts to describing the the IEAC signal as piecewise constant. The problem associated with this method is then to find appropriate interval boundaries. Another method is to assume that the model is not stationary, but that its second-order moment follows a linear path though a high-dimensional space. This approach means that the IEAC signal changes linearly with time; this type of model would have a projection onto its first principal component which was linear. Finally, we may assume that the time trajectory of the filter coefficients is non-linear; this approach would subsume ways of parameterizing (and thus smoothing) the IEAC signal such as using splines to fit the evolution of the autocorrelation parameters.

Ultimately, how to parameterize the IEAC signal must be left to domain-specific knowledge. Whether even linear interpolation between two sets of filter coefficients warrants the added complexity of having a filter (predictor) whose coefficients change as a function of time remains to be seen. Linear model parameterization of the IEAC, and its use in developing an associated predictor, will be taken up in a companion paper [2].

References

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[2] A. M. Rogers. (2003, May) Time-varying autoregressive modelling of audio and speech signals. [Online]. Available: http://www.arborrhythms.com